Exam - Statistics 2018/2019

Date: November 5, 2018
Time: 14.00-17.00
Place: ACLO station, Stationsplein 7-9, 9726AE Groningen
Progress code: WISTAT-07

Rules to follow:

- This is a closed book exam. Consultation of books and notes is **not** permitted.
- You can make us of a simple (non-programmable) calculator.
- Do not forget to write your name and student number onto each paper sheet.
- There are 5 exercises and the number of points per exercise are indicated within boxes. You can reach 90 points.
- For derivations include the relevant equation(s) and/or a short description.
- **We wish you success with the completion of the exam!**

START OF EXAM

1. **Geometric distribution.** **[25]**
   We have a random sample $X_1, \ldots, X_n$ from a geometric distribution with parameter $\theta \in (0,1)$. The pdf and the cdf of a geometric distribution with parameter θ are

   $$p_\theta(x) = \theta \cdot (1-\theta)^x \qquad (x \in \mathbb{N}_0)$$

   $$F_\theta(x) = 1 - (1-\theta)^{x+1} \qquad (x \in \mathbb{N}_0)$$

   The expectation of the geometric distribution is $\frac{1-\theta}{\theta}$. The variance of the geometric distribution is $\frac{1-\theta}{\theta^2}$.

   (a) [5] Derive the maximum likelihood estimator (MLE) of θ.
   Hint: Don't forget to show that this is really a maximum.

   (b) [5] Derive the methods of moments estimator (MOM) of θ, and compare the MOM with the MLE.

   (c) [5] Compute the expected Fisher information $I(\theta)$.

   Now assume that $n = 1$, so that we have only one single random variable X_1 in our random sample, and consider the test problem:

   $$H_0 : \theta = 0.4 \quad \text{versus} \quad H_1 : \theta = 0.2$$

   (d) [10] We want to perform the uniformly most powerful (UMP) test to the level α, where α should be as large as possible but at most 0.05. Derive the test statistic and the rejection region. Compute the power of this test at $\theta = 0.2$. Compute the p-value of the test, given the observation $x_1 = 9$.

2. **Unknown variance.** $\boxed{15}$
We have a random sample $X_1, \dots, X_n$ from a Gaussian distribution. The mean μ is known and the variance σ^2 is unknown. The pdf of a Gaussian distribution is

$$p(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\} \qquad (x \in \mathbb{R})$$

It can be shown that the statistic:

$$T(X_1, \dots, X_n) = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2}$$

is χ^2 (Chi-squared) distributed with n degrees of freedom. The variance of the χ^2 distribution is equal to $2n$. You can make use of these results.

(a) $\boxed{5}$ Derive the maximum likelihood estimator (MLE) of σ^2.
Hint: You do **not** have to show that this is really a maximum.

(b) $\boxed{5}$ Compute the mean squared error (MSE) of the MLE of σ^2. Is the MLE of σ^2 asymptotically consistent?
Hint: When computing the MSE, maybe you can somehow make use of the distribution of $T(X_1, \dots, X_n)$.

(c) $\boxed{5}$ Use the statistic $T(.)$ to construct a two-sided 90% confidence interval for the unknown variance σ^2. You can use generic symbols for the quantiles. E.g. let $q_{0.5}$ denote the 0.5-quantile of the χ^2 distribution.

3. **Simple regression.** $\boxed{20}$
Let $(Y_1, x_1), ..., (Y_n, x_n)$ be the data, where $Y_1, \dots, Y_n$ are independently and exponentially distributed random variables in the following way:

$$Y_i \sim EXP(\lambda x_i) \quad (i = 1, 2, \dots, n)$$

where $\lambda > 0$. The pdf of the i-th variable Y_i is then:

$$f_{Y_i}(y_i) = \lambda \cdot x_i \cdot e^{-\lambda \cdot x_i \cdot y_i} \quad (y_i \geq 0)$$

The known constants $x_1, \dots, x_n$ are strictly positive.

(a) $\boxed{5}$ Derive the maximum likelihood estimator (MLE) of λ.
Hint: Don't forget to show that this is really a maximum.

(b) $\boxed{5}$ Check whether the MLE is a sufficient statistic.

(c) $\boxed{10}$ Give the asymptotic distribution of the MLE, and then use the asymptotic distribution to derive an asymptotic 95% confidence interval for λ.
Hint: Use that the 0.975-quantile of the $\mathcal{N}(0, 1)$ is approximately 2.

4. **Estimator properties and tests.** **20**

Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter $\theta > 0$ and pdf:

$$p(x) = \frac{\theta^x}{x!} e^{-\theta} \qquad (x \in \mathbb{N}_0)$$

The mean and the variance of the Poisson distribution are both equal to θ.

(a) 5 Derive the maximum likelihood estimator (MLE) of θ.
Hint: Don't forget to show that this is really a maximum.

(b) 5 Derive the Cramer-Rao bound for the variance of unbiased estimators of θ.

(c) 5 Check whether $\hat{\theta}$ is unbiased and whether it attains the Cramer-Rao bound.

(d) 5 Given $n = 4$ and the four observations: $x_1 = 2$, $x_2 = 5$, $x_3 = 4$, $x_4 = 5$, and the test problem $H_0 : \theta \geq 7$ versus $H_1 : \theta < 7$. Construct the UMP test to the level α, where $\alpha \leq 0.05$ should be as large as possible.
Give the rejection region. Does the UMP test reject the null hypothesis?
HINT: The statistic $\sum_{i=1}^{n} X_i$ is Poisson distributed with parameter $n\theta$. A table with two quantiles of several Poisson distribution can be found in Table 1.

5. **Score confidence interval.** **10**

Consider an i.i.d. sample $X_1, \ldots, X_n$, denoted X, from a distribution F_θ, where $\theta \in \Theta \subset \mathbb{R}$ is an open set. Throughout this exercise you can assume that all required regulatory conditions are fulfilled.

(a) 5 Briefly show that $E[\frac{d}{d\theta} l_{X_1}(\theta)] = 0$ and $Var(\frac{d}{d\theta} l_{X_1}(\theta)) = I(\theta)$.

(b) 5 Use your result from part (a) in combination with the central limit theorem to show that the test statistic:

$$T(X) = \frac{\frac{d}{d\theta} l_X(\theta)}{\sqrt{n} \cdot \sqrt{I(\theta)}}$$

is asymptotically standard Gaussian $\mathcal{N}(0, 1)$ distributed.

λ	2	4	7	16	28
$q_{0.05,\lambda}$	0	1	3	10	20
$q_{0.95,\lambda}$	5	8	12	23	37

Table 1: **Poisson quantiles.** The entries for $q_{\alpha,\lambda}$ correspond to smallest integer x, such that $P(X \leq x) \geq \alpha$, where X has a Possion distribution with parameter λ.

END OF EXAM